The Cross Ratio

Invariance, Concyclicity, Symmetry, and Automorphisms of $\mathbb{D}$ and $\mathbb{H}$

Lecture Notes

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1 The Normalizing Möbius Map

Let $z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ be three distinct points.

Theorem

There is a unique Möbius map S with $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$.

Proof. If $z_2, z_3, z_4 \neq \infty$:

$$S(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3};$$

and if one of them is ∞:

$$z_2 = \infty : \ S(z) = \frac{z - z_3}{z - z_4}, \qquad z_3 = \infty : \ S(z) = \frac{z_2 - z_4}{z - z_4}, \qquad z_4 = \infty : \ S(z) = \frac{z - z_3}{z_2 - z_3}.$$

Uniqueness. If S_1 is another such map, then SS_1^{-1} fixes $0, 1, \infty$: writing $SS_1^{-1}(z) = \frac{az+b}{cz+d}$, fixing 0 gives $b = 0$, fixing ∞ gives $c = 0$, and fixing 1 gives $a = d$. So $SS_1^{-1} = \mathrm{id}$, i.e. $S = S_1$. □

Definition (cross ratio)

The **cross ratio** $(z_1, z_2, z_3, z_4) := S(z_1)$, where S is the Möbius map with $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$.

2 Invariance of the Cross Ratio

Theorem (Möbius invariance)

Let T be a Möbius map. Then for any four distinct $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$,

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4).$$

Proof. Let $Sz_2 = 1$, $Sz_3 = 0$, $Sz_4 = \infty$. Then $ST^{-1}(Tz_2) = 1$, $ST^{-1}(Tz_3) = 0$, $ST^{-1}(Tz_4) = \infty$, so ST^{-1} is *the* normalizing map for Tz_2, Tz_3, Tz_4. Hence

$$(Tz_1, Tz_2, Tz_3, Tz_4) = ST^{-1}(Tz_1) = S(z_1) = (z_1, z_2, z_3, z_4). \qquad \square$$

Theorem (cross ratio and concyclicity)

$(z_1, z_2, z_3, z_4) \in \mathbb{R}$ if and only if z_1, z_2, z_3, z_4 lie on the same circle or line.

Proof. Let S be the normalizing map for z_2, z_3, z_4 (so $Sz_2, Sz_3, Sz_4 = 1, 0, \infty$). By invariance,

$$(z_1, z_2, z_3, z_4) = (Sz_1, 1, 0, \infty) = Sz_1.$$

So

$$(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff Sz_1 \in \mathbb{R} \iff z_1 \text{ lies on the line/circle } S^{-1}(\mathbb{R} \cup \{\infty\}),$$

and since Möbius maps send lines-and-circles to lines-and-circles, $S^{-1}(\mathbb{R} \cup \{\infty\})$ is exactly the line or circle through z_2, z_3, z_4 (it contains them, as $Sz_2 = 1, Sz_3 = 0, Sz_4 = \infty \in \mathbb{R} \cup \{\infty\}$). □

3 Symmetry

Symmetry with respect to $\mathbb{R}$. The map $z \mapsto \bar{z}$ satisfies

$$(\bar{z}, 1, 0, \infty) = \overline{(z, 1, 0, \infty)}.$$

Definition (symmetric points)

Points z and z^* are **symmetric** with respect to the line or circle through z_2, z_3, z_4 if

$$(z, z_2, z_3, z_4) = \overline{(z^*, z_2, z_3, z_4)} \qquad (\text{equivalently } Sz = \overline{Sz^*}).$$

Theorem

This notion does not depend on the choice of z_2, z_3, z_4 on the given line or circle. Symmetry with respect to a **line** is the usual reflection. Symmetry with respect to a **circle** $|w - a| = r$ is given by **inversion**:

$$(z^* - a)\,\overline{(z - a)} = r^2.$$

Proof. *First*, if $z_2, z_3, z_4 \in \mathbb{R}$, the normalizing map S has real coefficients, so $S(\bar{z}) = \overline{S(z)}$: real coefficients commute with conjugation. If z_2, z_3, z_4 lie on a line, an affine map $Tz = \alpha z + \beta$ sends them into $\mathbb{R}$; since T preserves both cross ratio and (by the real case) symmetry, the symmetric point with respect to the line is the usual mirror image.

For a circle $|w - a| = r$: by translating and rescaling by the Möbius map $\varphi(z) := (z - a)/r$ – which sends the circle to the unit circle, and preserves cross ratio and symmetry – it suffices to treat $a = 0, r = 1$. Pick z_2, z_3, z_4 on the unit circle, so $\bar{z}_j = 1/z_j$ for $j = 2, 3, 4$. For $w := 1/\bar{z}$,

$$\overline{(w, z_2, z_3, z_4)} = (\bar{w}, \bar{z}_2, \bar{z}_3, \bar{z}_4) = \left(\tfrac{1}{z}, \tfrac{1}{z_2}, \tfrac{1}{z_3}, \tfrac{1}{z_4}\right) = (z, z_2, z_3, z_4),$$

where the last equality is Möbius invariance applied to the map $\zeta \mapsto 1/\zeta$. So $z^* = 1/\bar{z}$ is the symmetric point when $a = 0, r = 1$, i.e. $z^*\bar{z} = 1 = r^2$. Undoing the rescaling φ gives the general formula $(z^* - a)\overline{(z - a)} = r^2$. □

Remark. With respect to $\{|z - a| = r\}$, the symmetric point of a is ∞. *Verification:* take $z_2 = a + r$, $z_3 = a - r$, $z_4 = a + ir$ on the circle. Then

$$(\infty, z_2, z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3} = \frac{r - ir}{2r} = \frac{1 - i}{2},$$

$$(a, z_2, z_3, z_4) = \frac{a - z_3}{a - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} = \frac{r}{-ir} \cdot \frac{1 - i}{2} = i \cdot \frac{1 - i}{2} = \frac{1 + i}{2} = \overline{(\infty, z_2, z_3, z_4)},$$

confirming a and ∞ are symmetric with respect to the circle.

Corollary

If T is Möbius and z, z^* are symmetric with respect to a circle or line ℓ, then Tz, Tz^* are symmetric with respect to $T\ell$.

Proof. Let $z_2, z_3, z_4 \in \ell$. By definition and invariance,

$$(Tz, Tz_2, Tz_3, Tz_4) = (z, z_2, z_3, z_4) = \overline{(z^*, z_2, z_3, z_4)} = \overline{(Tz^*, Tz_2, Tz_3, Tz_4)},$$

so Tz, Tz^* are symmetric with respect to the circle/line through Tz_2, Tz_3, Tz_4, which is $T\ell$. □

4 Automorphisms of the Disc and the Half-Plane

Theorem

Let T be Möbius with $T(\mathbb{D}) = \mathbb{D}$ (where $\mathbb{D} = B(0,1)$). Then

$$Tz = e^{i\theta}\,\frac{z-a}{1-\bar{a}z} \qquad \text{for some } a \in \mathbb{D},\ \theta \in \mathbb{R}.$$

Proof. *Preliminary observation.* If $Tz = e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$, then for $|z| = 1$, writing $1 - \bar{a}z = z(\bar{z} - \bar{a})$,

$$|Tz| = \frac{|z-a|}{|z||\bar{z}-\bar{a}|} = \frac{|z-a|}{|z-a|} = 1,$$

so such a map sends the unit circle to itself, and (since $T(0) = -e^{i\theta}a \in \mathbb{D}$) sends $\mathbb{D}$ to $\mathbb{D}$.

Construction. Conversely, given T Möbius with $T(\mathbb{D}) = \mathbb{D}$, let $a := T^{-1}(0) \in \mathbb{D}$. Since $T(\mathbb{D}) = \mathbb{D}$ forces T to send the unit circle to itself, T preserves symmetry with respect to it; the symmetric point of a is $1/\bar{a}$ (by the Remark above), and the symmetric point of $T(a) = 0$ is ∞ (center $\leftrightarrow \infty$). So

$$T(1/\bar{a}) = \infty.$$

A Möbius map with $T(a) = 0$, $T(1/\bar{a}) = \infty$ has the form

$$Tz = c \cdot \frac{z-a}{z - 1/\bar{a}} = c\bar{a} \cdot \frac{z-a}{\bar{a}z - 1} =: d \cdot \frac{z-a}{1-\bar{a}z}.$$

Since $|z| = 1 \implies |Tz| = 1$, taking $z = 1$ gives $|d|\dfrac{|1-a|}{|1-\bar{a}|} = 1$; as $|1-a| = |1-\bar{a}|$ always, $|d| = 1$, so $d = e^{i\theta}$ for some $\theta \in \mathbb{R}$. ■

Theorem

Let $\mathbb{H} = \{\operatorname{Im} z > 0\}$. Then $T(\mathbb{H}) = \mathbb{H}$ if and only if

$$Tz = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

Proof: very similar, left as an exercise.